

# A Proof of Irrationality of $\pi$ by Contraposition\*

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## Abstract

We point out that the proof of irrationality of  $\pi$  by Niven can be modified to a proof by contraposition. As a warm-up, we also give a proof of irrationality of  $\sqrt{2}$  and  $\sqrt{3}$  in a similar way.

## 1 Introduction

Though contradiction and contraposition are logically equivalent, it is not always clear how to translate a proof by contradiction to the one by contraposition. In this note we give a proof of irrationality of  $\pi$  by contraposition, which is obtained by modifying Niven's famous proof [1]. The strategy of our proof is inspired by [2, Theorem 2.3], an irrationality proof of  $\sqrt{2}$  by contradiction, with set-theoretic translation. Our proof of irrationality of  $\pi$  is given in Section 3, which follows the section that is devoted to the irrationality of  $\sqrt{2}$  and  $\sqrt{3}$  to a warm-up of our strategy.

## 2 Proof of Irrationality of $\sqrt{2}$ and $\sqrt{3}$

For a given  $n \in \mathbb{N}$ , consider the set  $P_n$  of rational numbers  $r$  with  $r^2 = n$ . Then  $n$  is either the square of an odd number or divided by 4 if  $P_n$  is not the empty set  $\emptyset$ , since the square of any element  $r \in P_n$  is expressed as  $4^{p-q} (k/l)^2$ , where  $r = (2^p k) / (2^q l)$  for some odd numbers  $k$  and  $l$ . This implies  $P_2 = \emptyset$  and  $P_3 = \emptyset$ .

## 3 Proof of Irrationality of $\pi$

The main idea of the proof in Section 2 is to take an indexed family of sets that are given as solution sets of equations, and then show that sets with particular

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indices are empty by contraposition. We apply this idea to Niven's proof [1] of irrationality of  $\pi$ .

Since we use the differentiation of trigonometric functions,  $\pi$  is supposed to be defined analytically; for example, it is the smallest positive number for  $\sin x = 0$ , where the trigonometric function  $\sin x$  is defined as the infinite series

$$\sin x := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

For a given rational number  $r = a/b \in \mathbb{Q}$ , where  $a \in \mathbb{Z}$  and  $b \in \mathbb{N}$ , and a positive integer  $n \in \mathbb{N}$ , define a function  $F_n$  by

$$F_n(x) := \sum_{j=0}^n (-1)^j f_n^{(2j)}(x),$$

where  $f_n(x) := (b^n/n!) x^n (r-x)^n$ . The following properties are satisfied for any  $r \in \mathbb{Q}$ :

- (1)  $F_n(0) = F_n(r)$  for any  $n \in \mathbb{N}$ ;
- (2)  $F_n(0) \in \mathbb{Z}$  for any  $n \in \mathbb{N}$ ;
- (3) for any  $N \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  such that  $n \geq N$  and  $F_n(0) \neq 0$ .

Properties (1) and (2) are substantially shown in [1]. The proof of (3) is given later.

Now, let  $Q_k$  be the set of positive rational numbers  $r$  satisfying  $\sin r = 0$  and  $\cos r = k$ . By the equality  $\sin^2 r + \cos^2 r = 1$ , the number  $k$  is either 1 or  $-1$ . What we need to show is that  $\pi \notin Q_{-1}$ . It is enough to show that  $Q_{-1} = \emptyset$ , which is done by its contrapositive, i.e., from now on we show that  $k$  must be 1 if  $Q_k \neq \emptyset$ .

Since  $Q_k \neq \emptyset$ , we can take an element  $r = a/b \in Q_k$ . For such  $r$  and for any  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \int_0^r f_n(x) \sin x \, dx &= \left[ F_n^{(1)}(x) \sin x - F_n(x) \cos x \right]_0^r \\ &= (1 - k) F_n(0) \end{aligned}$$

by (1).

Since  $-1 \leq \sin x \leq 1$  and  $0 < f_n(x) < (b r^2)^n / n!$  for any  $x \in (0, r)$  and for any  $n \in \mathbb{N}$ , we have

$$-\frac{(b r^2)^n}{n!} < f_n(x) \sin x < \frac{(b r^2)^n}{n!}$$

on  $(0, r)$ . Since  $r > 0$ , integrating it on  $[0, r]$  and we have

$$-\frac{r (b r^2)^n}{n!} = -\int_0^r \frac{(b r^2)^n}{n!} \, dx < \int_0^r f_n(x) \sin x \, dx < \int_0^r \frac{(b r^2)^n}{n!} \, dx = \frac{r (b r^2)^n}{n!}.$$

This inequality implies

$$-1 < \int_0^r f_n(x) \sin x \, dx = (1 - k) F_n(0) < 1$$

for any sufficiently large  $n \in \mathbb{N}$ . Since  $(1 - k) F_n(0)$  is an integer by (2), it must be 0. Take  $n$  to be the one given in (3) and we have  $k = 1$ .

Lastly, we prove Property (3). For a given  $N \in \mathbb{N}$ , take an odd prime number  $p$  with  $p > N$ , and take  $n$  to be  $p - 1$ .

Since  $n$  is even, the expression of  $F_n(0)$  is  $b^n$  times, unlike odd  $n$ , a monic polynomial with integer coefficients up to sign as follows:

$$F_n(0) = b^n \sum_{i=0}^n a_{n,i} r^i,$$

where

$$a_{n,i} := \begin{cases} 0, & \text{if } i \text{ is odd,} \\ (-1)^{\frac{i}{2}} \binom{n}{n-i} \frac{(2n-i)!}{n!}, & \text{if } i \text{ is even.} \end{cases}$$

Showing the irreducibility of  $F_n(0)/b^n$  in the polynomial ring in  $r$  over  $\mathbb{Q}$  is sufficient to prove (3). By Eisenstein's irreducibility criterion, it is enough to see the following facts:

- (i)  $p$  divides each  $a_{n,i}$  for  $i = 0, 1, \dots, n - 1$ ;
- (ii)  $p^2$  does not divide  $a_{n,0}$ .

Since  $a_{n,i} = 0$  for odd  $i$ , we assume that  $i = 0, 2, \dots, n - 2$  in the following. For such  $i$ , the integer  $p = n + 1$  divides  $(2n - i)!/n!$ . We have thus proved (i). For (ii), since  $2n < 2n + 2 = 2p$ , only the term  $n + 1$  in the expression  $a_{n,0} = (2n)!/n! = 2n(2n - 1) \cdots (n + 1)$  as a product of integers is divided by the prime number  $p = n + 1$ .  $\square$

## References

- [1] I. Niven, *A simple proof that  $\pi$  is irrational*, Bull. Amer. Math. Soc. **53** (1947), 509.
- [2] H. Tasaki, *Mathematics—for those who learn and enjoy physics—*, written in Japanese (2013), <http://www.gakushuin.ac.jp/~881791/mathbook/>.